A Proof that the Divergence of a Surface Normal Is Equal to the Sum of the Principal Curvatures

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Motivation

In solving problems that include an interface it is often important to calculate the change in pressure, ΔP, across the interface. The Young-Laplace equation tells us that when the interface is modelled as a curve

$$\Delta P = \gamma \nabla \cdot \mathbf{n} = \gamma \kappa$$

and when the interface is modelled as a surface

$$\Delta P = \gamma \nabla \cdot \mathbf{n} = \gamma(\kappa_1 + \kappa_2).$$

Here, γ is the surface tension, $\mathbf{n}$ is the normal to the curve or surface, and κ and κ_1 and κ_2 are curvatures for the curve and surface. The curvatures can also be written as the reciprocal of the radii of curvature. Considering the different forms for the Young-Laplace equation leads to the question of how one might show that the two formulations are equivalent. In other words, we would like to show mathematically that

$$\nabla \cdot \mathbf{n} = \kappa_1 + \kappa_2$$

as an alternative to deriving the Young-Laplace equation for the two separate forms.

Statement of problem

Let a curve, $h : \mathbf{R} \to \mathbf{R}$, in two dimensions and a surface, $f : \mathbf{R}^2 \to \mathbf{R}$ in three dimensions be defined by $y = h(x)$ and $z = f(x, y)$ respectively. Let each of these functions have continuous second derivatives. Then in two dimensions, the divergence of the normal to the curve is equal to the curvature. That is,

$$\nabla \cdot \mathbf{n} = \kappa.$$

In three dimensions, the divergence of the normal to the surface is equal to the sum of the principal curvatures. The principal curvatures are defined as the maximum and minimum of all two dimensional curvatures found by considering the intersection of the surface and a plane that contains the surface normal. Or, more simply,

$$\nabla \cdot \mathbf{n} = \kappa_1 + \kappa_2.$$

Proof of statement

Case 1: Two dimensions

Define the normal and calculate its divergence in the following way.

$$\begin{aligned}
F(x,y) &= y - h(x) \\
\mathbf{n} &= \frac{\nabla F(x,y)}{|\nabla F(x,y)|} = \frac{e_y - h_x(x)e_x}{\sqrt{1+h_x(x)^2}} \\
\nabla \cdot \mathbf{n} &= -h_{xx}(x)(1+h_x(x)^2)^{-1/2} + h_x(x)(1+h_x(x)^2)^{-3/2} h_x(x)h_{xx}(x) \\
&= \frac{-h_{xx}(x) - h_{xx}(x)h_x(x)^2 + h_x(x)^2 h_{xx}(x)}{(1+h_x(x)^2)^{3/2}} \\
&= \frac{-h_{xx}(x)}{(1+h_x(x)^2)^{3/2}}
\end{aligned}$$

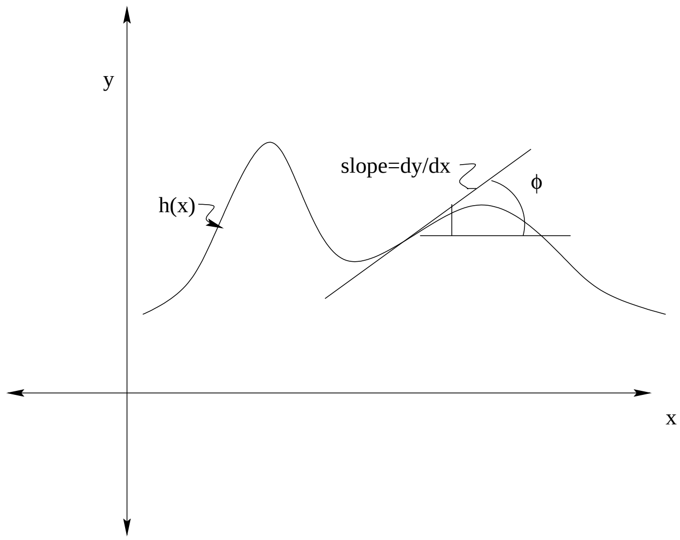
Note that the computations utilize the outward normal.

Most calculus books define the curvature of a curve as a positive number.

$$\kappa = \left|\frac{d\phi}{ds}\right|$$

Here ϕ is the angle between the horizontal and the tangent to the curve. We differentiate ϕ with respect to the arc length, s, to define the curvature. For our purposes we would like to have a signed curvature so we begin without the absolute values.

$$\begin{aligned}
\kappa &= \frac{d\phi}{ds} \\
\frac{d\phi}{dx} &= \frac{d\phi}{ds}\frac{ds}{dx} \\
\phi &= \tan^{-1}\frac{dy}{dx} = \tan^{-1} h_x(x) \\
\frac{d\phi}{dx} &= \frac{h_{xx}(x)}{1+h_x(x)^2} \\
\frac{ds}{dx} &= \frac{d}{dx}\int_0^x \sqrt{1+h_x(\xi)^2}\,d\xi = \sqrt{1+h_x(x)^2} \\
\kappa = \frac{d\phi/dx}{ds/dx} &= \frac{h_{xx}(x)}{1+h_x(x)^2}\frac{1}{\sqrt{1+h_x(x)^2}} = \frac{h_{xx}(x)}{(1+h_x(x)^2)^{3/2}}
\end{aligned}$$



The proof is complete except for a difference in sign. However, in considering the normal to a surface, there is a choice to be made on whether to consider the inward or outward normal. As noted above we are interested in the outward normal. Likewise, we must choose a sign for the curvature. By convention, we will choose postive curvature to correspond to a curve that is concave down. For a curve that is concave down

$$h_{xx} \leq 0.$$

Therefore,

$$\kappa = \frac{-h_{xx}}{(1 + h_x^2)^{3/2}}$$

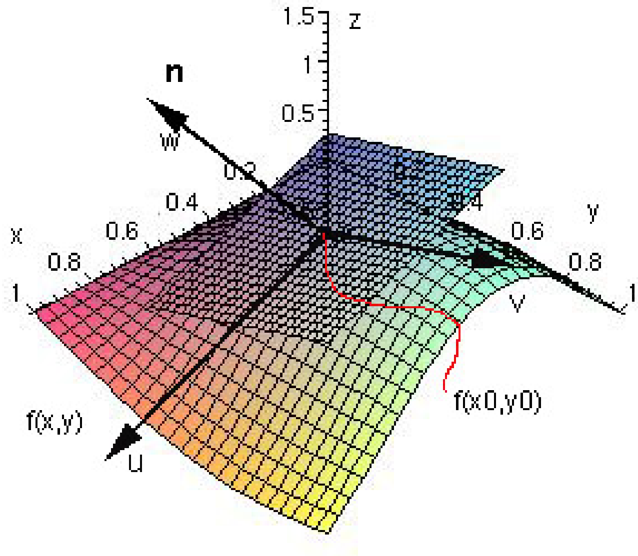
gives the desired sign and completes the proof.

Case 2: Three Dimensions

In three dimensions, there is more than one curvature at a point x_0. In fact, there are infinitely many planes that contain the normal to the surface at x_0 and the interection of each of these planes with the surface defines a two dimensional curve and a corresponding curvature. To find the principal curvatures, the maximum and minimum of the set of curvatures described above, we would expect to calculate the curvature in a particular direction and then minimize and maximize over direction. The algebra needed to accomplish this directly is quite messy. In order to simplify this algebraic computation we will first note that the identity,

$$\nabla \cdot \mathbf{n} = \frac{1}{\kappa_1} + \frac{1}{\kappa_2}$$

in Cartesian coordinates does not depend on the location of the coordinate axes. Therefore we can rotate and translate the x-y-z-axes as we please. In particular, we first fix a point where we will calculate the curvatures and the divergence of the normal, $x_0 = (x, y, f(x, y))$. Next we rotate and translate the coordinate axes so that the $x-y$ plane corresponds to the plane tangent to f at x_0. We will call this new set of axes, u, v, w as shown in the following picture.



Note that this new set of axes has the following properties:

$$\begin{aligned} f(0,0) &= 0 \\ f_u(0,0) &= 0 \\ f_v(0,0) &= 0 \end{aligned}$$

where $(0, 0, 0)$ in the u-v-w system corresponds to x_0 in the x-y-z system. This set of axes also has the property that the normal to the surface at 0 coincides with the w-axis. Because of this, any plane that contains the normal also contains the line $v = ru$ for some r and this line is perpendicular to the normal. This will considerably simplify the computations for the rest of the proof.

First, we will prove that the directions corresponding to the principal curvatures are perpendicular to each other. The u-v-w system allows us to reduce our problem to two dimensions very easily. We simply choose a direction in which we want to calculate the curvature and substitute $v = ru$ as mentioned above with r corresponding to the chosen direction. This gives a curve that lies in a plane containing the normal to the surface. In particular

$$w = f(u, ru).$$

Calculating the curvature for this curve, we must be careful to differentiate f in the direction corresponding to r. We can parameterize f in this direction with a variable t given by the following.

$$
\begin{aligned}
t^2 &= u^2 + v^2 = u^2(1+r^2) \\
u^2 &= \frac{t^2}{r^2+1} \\
\frac{du}{dt} &= \frac{1}{\sqrt{r^2+1}} \\
w = f(u, ru) &= f(t/\sqrt{1+r^2}, rt/\sqrt{1+r^2}).
\end{aligned}
$$

Calculate the curvature in the r direction.

$$
\begin{aligned}
\kappa_r &= \frac{d\phi}{ds} \\
\frac{d\phi}{dt} &= \frac{d\phi}{ds}\frac{ds}{dt} \\
\phi = \tan^{-1}\frac{dw}{dt} &= \tan^{-1}\frac{f_u(u,ru) + kf_v(u,ru)}{\sqrt{1+r^2}} \\
\frac{d\phi}{dt} &= \frac{1}{1+\frac{(f_u(u,ru)+rf_v(u,ru))^2}{1+r^2}}\frac{f_{uu}(u,ru) + 2rf_{uv}(u,ru) + r^2 f_{vv}(u,ru)}{1+r^2} \\
&= \frac{f_{uu}(u,ru) + 2rf_{uv}(u,ru) + r^2 f_{vv}(u,ru)}{1+r^2+(f_u(u,ru)+rf_v(u,ru))^2} \\
\frac{ds}{dt} &= \frac{d}{dt}\int_0^u \sqrt{1+\frac{(f_u(\xi,r\xi)+rf_v(\xi,r\xi))^2}{1+r^2}}\sqrt{1+r^2}d\xi = \sqrt{1+\frac{(f_u(u,ru)+rf_v(u,ru))^2}{1+r^2}} \\
\kappa_r = \frac{d\phi/dt}{ds/dt} &= \frac{f_{uu}(u,ru) + 2rf_{uv}(u,ru) + r^2 f_{vv}(u,ru)}{1+r^2+(f_u(u,ru)+rf_v(u,ru))^2}\frac{\sqrt{1+r^2}}{\sqrt{1+r^2+(f_u(u,ru)+rf_v(u,ru))^2}}
\end{aligned}
$$

Substituting $t = 0$ and using the fact that $f_u(0,0) = f_v(0,0) = 0$, we obtain the curvature of the surface at 0 in the r direction.

$$\kappa_r = \frac{f_{uu}(0,0) + 2rf_{uv}(0,0) + r^2 f_{vv}(0,0)}{1+r^2}$$

For the following calculations we will set

$$a = f_{uu}(0,0)$$

$$b = 2f_{uv}(0,0)$$

$$c = f_{vv}(0,0)$$

which are constants. The next step is to find the critical points of the curvature at 0 over the direction, r.

$$\begin{aligned}
\frac{d\kappa_r}{dr} &= \frac{d}{dr}\left(\frac{a+br+cr^2}{1+r^2}\right) = \frac{b+2cr}{r^2+1} - \frac{(a+br+cr^2)(2r)}{(r^2+1)^2} \\
&= \frac{br^2+2cr^3+b+2cr-2ar-2br^2-2cr^3}{(r^2+1)^2} \\
&= \frac{-br^2+(2c-2a)r+b}{(r^2+1)^2}
\end{aligned}$$

Setting this equal to zero, we obtain a quadratic polynomial in r

$$0 = -br^2 + (2c-2a)r + b$$

Suppose r_0 satisfies this equation. Then,

$$\frac{-b}{r_0^2} - \frac{(2c-2a)}{r_0} + b = \frac{-1}{r_0^2}(b + (2c-2a)r_0 - br_0^2) = 0$$

shows that $-1/r_0$ satisfies the equation also, proving that the directions of the principal curvatures are perpendicular to each other. Solving for these roots

$$r = \frac{(2a-2c) \pm \sqrt{(2a-2c)^2 + 4b^2}}{-2b}$$

$$= \frac{c-a}{b} \pm \sqrt{\left(\frac{a-c}{b}\right)^2 + 1}$$

Therefore the directions of maximum and minimum curvature on the u-v-w axes at 0 are given by

$$r = \frac{f_{vv}(0,0) - f_{uu}(0,0)}{f_{uv}(0,0)} \pm \sqrt{\left(\frac{f_{uu}(0,0) - f_{vv}(0,0)}{f_{uv}(0,0)}\right)^2 + 1}$$

evaluated at zero. The sum of principal curvatures which are given by directions, $r, -1/r$ is

$$\begin{aligned}
\kappa_1 + \kappa_2 &= \frac{a+br+cr^2}{r^2+1} + \frac{a-b/r+c/r^2}{1/r^2+1} \\
&= \frac{a+br+cr^2+ar^2-br+c}{r^2+1} \\
&= \frac{(a+c)(r^2+1)}{r^2+1} \\
&= a+c \\
&= f_{uu}(0,0) + f_{vv}(0,0)
\end{aligned}$$

As in two dimensions, we must choose a sign for the curvatures that corresponds to the chosen convention that positive curvature corresponds to a curve that is concave down. Therefore,

$$\kappa_1 + \kappa_2 = -f_{uu}(0,0) - f_{vv}(0,0).$$

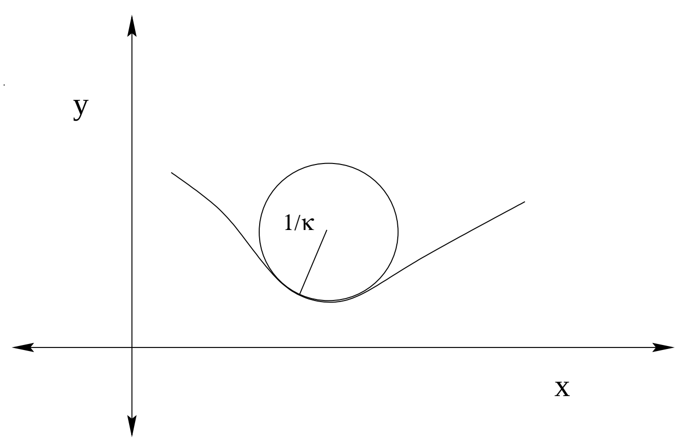
Now we need only calculate the divergence of the normal in the u-v-w coordinate system at zero.

$$\begin{aligned}
\nabla \cdot \mathbf{n} &= \nabla \cdot \left(\frac{-f_u \mathbf{e_u} - f_v \mathbf{e_v} + \mathbf{e_w}}{\sqrt{1 + f_u^2 + f_v^2}} \right) \\
&= \frac{-f_{uu} - f_{vv}}{\sqrt{1 + f_u^2 + f_v^2}} + \frac{f_u(f_u f_{uu} + f_v f_{uv}) + f_v(f_u f_{uv} + f_v f_{vv})}{(1 + f_u^2 + f_v^2)^{\frac{3}{2}}} \\
\nabla \cdot \mathbf{n} &= -f_{uu}(0,0) - f_{vv}(0,0)
\end{aligned}$$

once we evaluate at zero. This completes the proof.

Concluding remarks

In the above proof, we have not used radii of curvature. The radius of curvature at a point, p, is by definition the reciprocal of the curvature at p. Because a circle of radius R has curvature $1/R$, the circle tangent to a curve at p with radius $1/\kappa$ has the same curvature as the curve and the same first derivative at p. Since the curvature depends only on the first and second derivatives, this particular circle must also have the same second derivative at p. From this we can see that the radius of curvature generates a circle that best fits the curve at p.



The proof mentions that the divergence of the normal and the curvatures do not change with translations and rotations of the coordinate axes. That the curvature is invariant modulo the sign is easy to see when one considers the radius of curvature. No matter how the curve is oriented with respect to the coordinate axes, the same size circle will best fit the curve at a particular point. The curvature is equal to the reciprocal of the radius of this circle and therefore does not vary. The divergence of the normal describes the net rate of change of the normal which is also invariant with respect to the coordinate axes.

Lastly, note that the above proof can be extended easily to curves and surfaces that are defined implicitly by functions of the form $f(x, y, z) = c$, or parametrically where x, y, and z are functions of two other variables. Also, for polar or spherical coordinates, the statement may be proved in a similar fashion with the correct definition of divergence and curvature. However, the transformation of these coordinates to the Cartesian system automatically generalizes the above proof.

References

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http://mathpages.com/rr/s5-03/5-03.htm.

Stewart, James. *Calculus*. Brooks-Cole: Pacific Grove, 1991.