Final Project Report

DEFORMATION OF A 3-DIMENSIONAL RISING BUBBLE

Submitted to:

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Introduction

Characterization of bubble rise is of importance to the design of mass transfer operations, such as bubble columns. The overall mass transfer is affected by the bubble size, pressure inside the gas phase, interaction between bubbles, rise velocity, and trajectory. Single gas bubbles rising through a liquid have been studied extensively, and it has been found that when the bubbles are very small, surface tension, which is predominant over the inertial force and the buoyant force, makes the bubbles spherical and they tend to preserve the spherical shape as long as their rising velocity, thus Reynolds number, remains small. In most practical circumstances, all three factors inertia effect, viscosity, and surface tension should be considered in that the bubbles are not spherical in shape and they move in a oscillatory manner.

The objective of this project is to quantify the deformation of rising bubble in infinite medium with constant velocity. The velocity potential is used to describe the flow past a 3-dimensional bubble, which of the interface is considered as a free surface where ‘slip’ occurs with zero shear stress. It is major advantage of the potential function analysis that the viscosity effect can be excluded with this ‘slip’ boundary condition even when the Reynolds number is not sufficiently high.

With the pressure difference calculated from the velocity potential, Young-Laplace equation is used for determining the shape of the free surface.

Model Description

For the stream function analysis to be valid, the liquid flow past a bubble is assumed to be irrotational, and the vortex inside the bubble induced by the outer liquid flow is ignored, i.e. there is no shear stress on the surface, thus no pressure gradient developed in the gas phase.

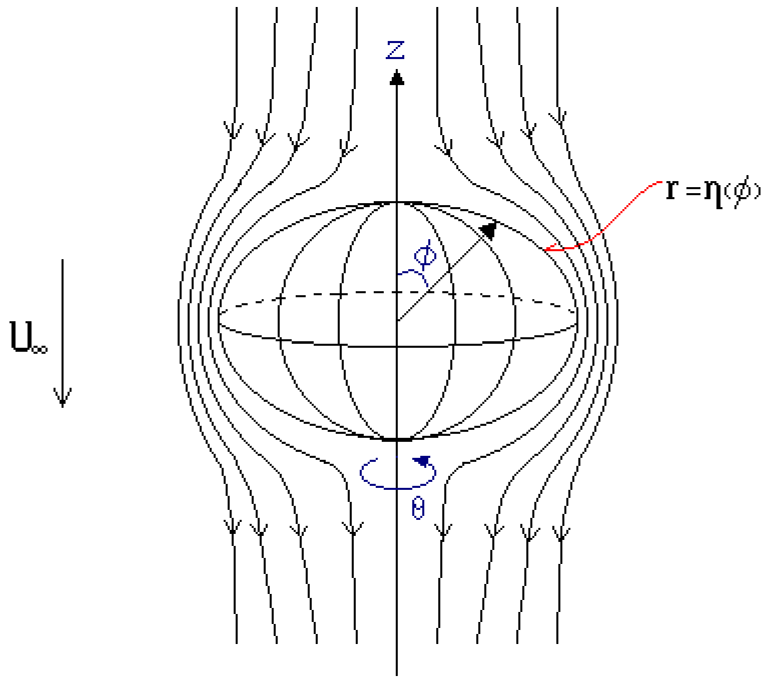

Figure 1. Potential flow past a deformed bubble

The description and computation of an irrotational flow is simplified substantially by introducing the velocity potential Φ defined in terms of the equation

$$\nabla\Phi = \vec{u} \qquad (1)$$

where $\vec{u}$ is the velocity vector. The continuity equation requires that, when the fluid is incompressible, the velocity potential Φ be harmonic function

$$\nabla^2\Phi = 0 \qquad (2)$$

The boundary conditions to be satisfied in the problem of a flow due to a moving body to are

$$\vec{n}\cdot\nabla\Phi = \vec{n}\cdot\vec{u}_\infty \qquad \text{at the free surface} \qquad (3)$$

$$\Phi \to C(const) \qquad \text{as} \quad r \to \infty \tag{4}$$

where $\vec{n}$ is unit normal vector of the free surface, and $\vec{u}_\infty$ is the terminal velocity vector of rising bubble.

Instead of solving the partial differential equation (2) of the harmonic function, the boundary condition (3) is used with the assumption: The tangential derivative of the velocity potential is negligible compared to the normal derivative of the potential.

The resulting velocity field generated by the moving bubble without slip on the surface can be converted to the pressure build-up on the free surface. The liquid pressure P_l is obtained from the Bernoulli equation

$$P_o = P_l + \frac{1}{2}\rho\left|\vec{u}\right|^2 = P_l + \frac{1}{2}\rho\left|\nabla\Phi\right|^2 \tag{5}$$

where P_0 is static pressure of the liquid on the free surface, and ρ is the density of liquid. When P_b denotes the uniform pressure inside the bubble, the pressure difference across the interface is equal to:

$$\begin{aligned}\Delta P &= P_b - P_o \\ &= P_b - P_{ls} - \frac{1}{2}\rho\left|\nabla\Phi\right|_s^2\end{aligned} \tag{6}$$

where subscript *s* denotes the quantity evaluated at the surface.

The pressure difference across the surface also be expressed by Young-Laplace equation

$$\Delta P = \gamma(\nabla \cdot \vec{n}) \tag{7}$$

Equating (6) and (7) yields the governing equation determining the free surface

$$P_b - P_{ls} - \frac{1}{2}\rho\left|\nabla\Phi\right|_s^2 = \gamma(\nabla \cdot \vec{n}) \tag{8}$$

With the functional representation of the free surface $r = \eta(\phi)$, the left hand side and the right hand side of the equation (8) becomes respectively:

(See **Appendix A.** for detail calculation)

$$(P_b - P_{ls}) - \frac{1}{2}\rho U_\infty^2 \left[\frac{\cos\phi + \sin\phi \frac{1}{\eta}\frac{\partial \eta}{\partial \phi}}{1 + \frac{1}{\eta^2}\left(\frac{\partial \eta}{\partial \phi}\right)^2} \right]^2 \left(1 + \frac{1}{\eta^2}\left(\frac{\partial \eta}{\partial \phi}\right)^2 \right)$$

$$= \frac{\gamma}{\eta} \frac{\left[2 + \frac{3}{\eta^2}\left(\frac{\partial \eta}{\partial \phi}\right)^2 - \cot\phi \left(\frac{1}{\eta}\left(\frac{\partial \eta}{\partial \phi}\right) + \frac{1}{\eta^3}\left(\frac{\partial \eta}{\partial \phi}\right)^3 \right) - \frac{1}{\eta}\frac{\partial^2 \eta}{\partial \phi^2} \right]}{\left[1 + \frac{1}{\eta^2}\left(\frac{\partial \eta}{\partial \phi}\right)^2 \right]^{3/2}} \qquad (9)$$

If the radius of a static bubble is set equal to R,

$$R = \frac{2\gamma}{P_b - P_{ls}} \qquad (10)$$

then the Webber number is defined as:

$$We = \frac{\rho U_\infty^2 R}{\gamma} \qquad (11)$$

The 2nd order ordinary differential equation (9) is reduced to a much simpler form by incorporating y which is defined as:

$$y = \ln\left[\frac{\eta}{R}\right] \qquad (12)$$

Finally, the differential equation to be solved is rearranged as:

$$y'' = 2 + 2y'^2 - \cot\phi\left[y' + y'^3\right] - 2\exp(y)\cdot\left[1 + y'^2\right]^{3/2}\left[1 - \frac{We}{4}\frac{(\cos\phi + y'\sin\phi)^2}{1 + y'^2}\right] \qquad (13)$$

with boundary conditions

$$y' = 0 \qquad \text{at} \quad \phi = 0 \quad \text{and} \quad \phi = \frac{\pi}{2} \qquad (14)$$

The solution to the non-linear 2nd order differential equation (13) is sought by numerical method using the Runge-Kutta-Nystrom algorithm for the several Webber numbers.

(See **Appendix B.** for detail)

Calculation Result

The numerical solutions are represented as a function of angle ϕ.

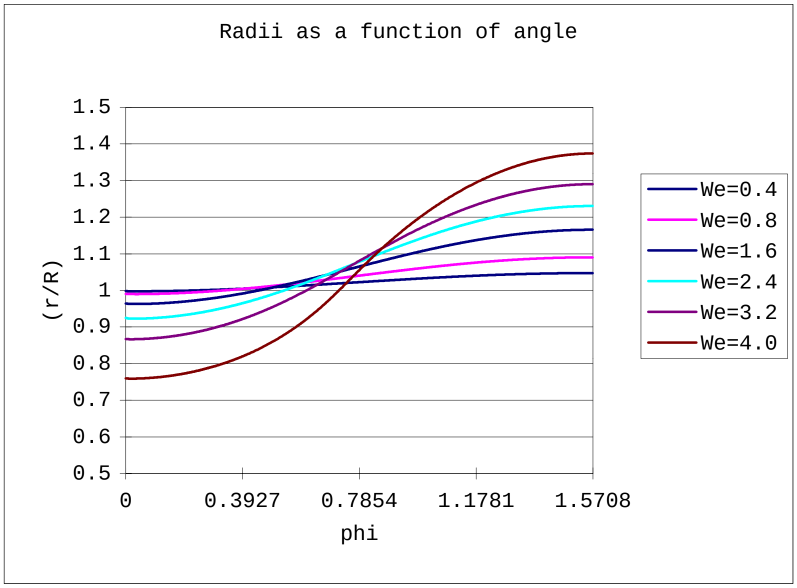

Figure 2. Numerical solution of $r = \eta(\phi)$ for several values of We

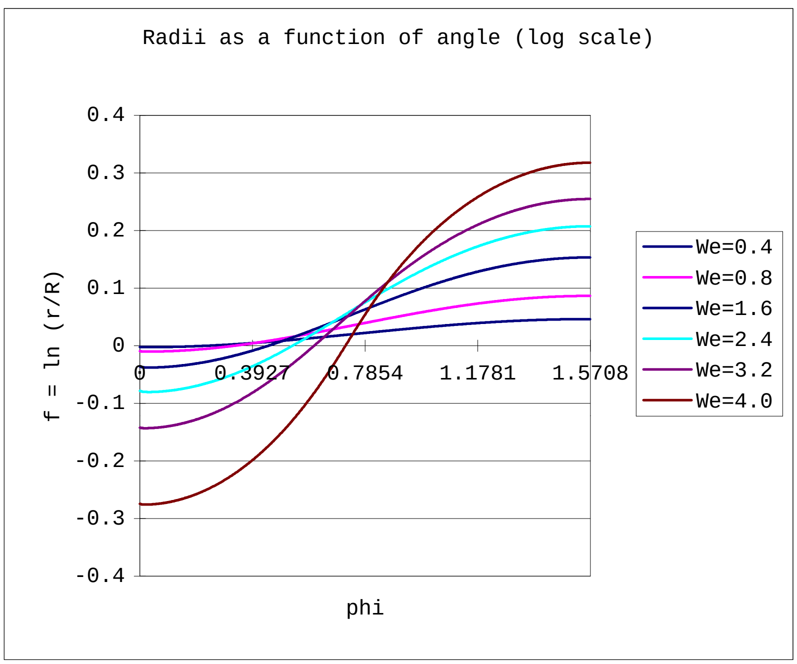

Figure 3. Numerical solution of $f = \ln\left[\dfrac{\eta(\phi)}{R}\right]$ for several values of We

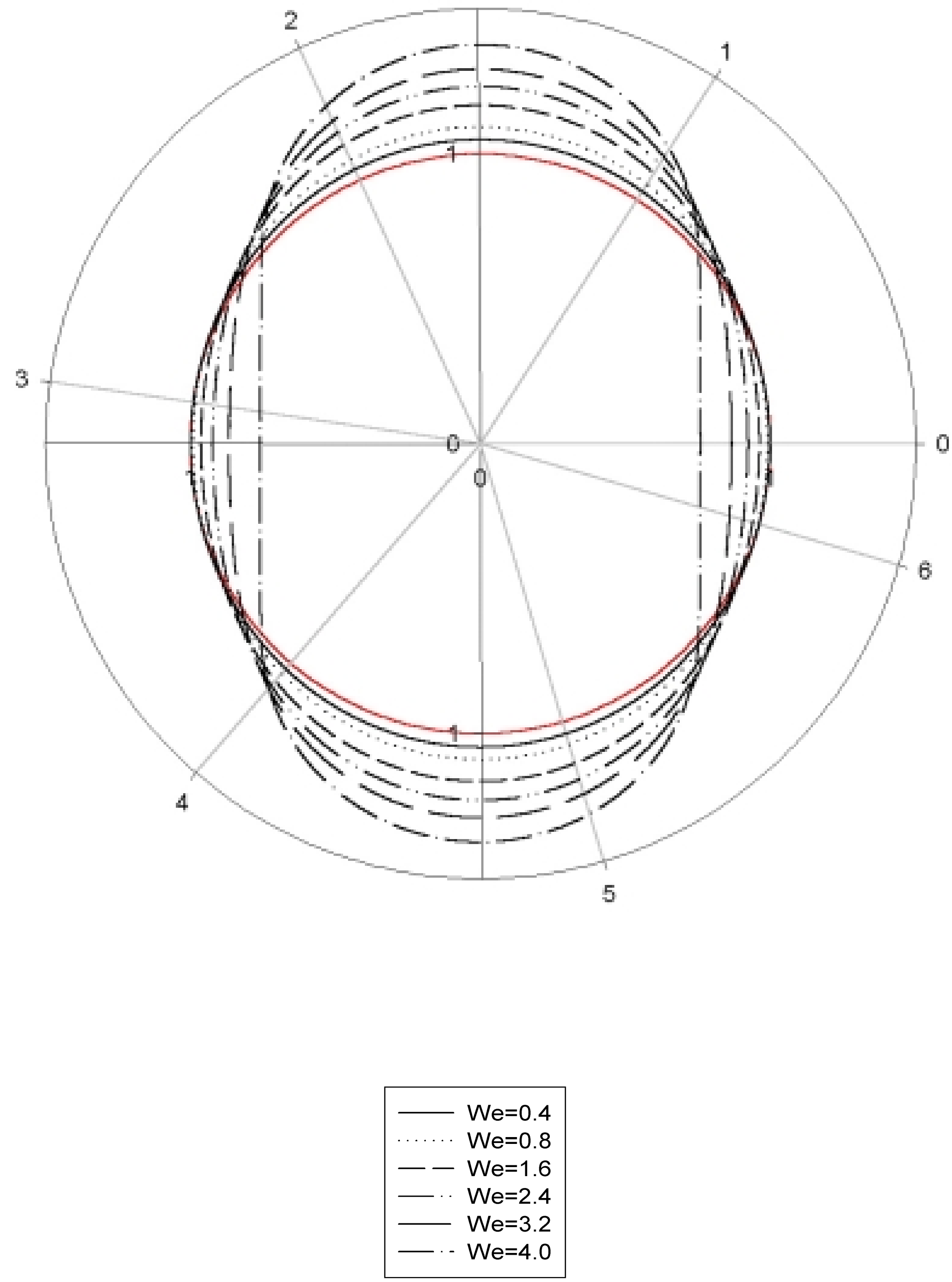

Figure 4. Bubble Deformation on Polar Plot

Conclusion

The calculation shows that the oblation of a bubble increases as the size of bubble, thus the Webber number increases. However, the shape of a rising bubble with We=4.0 which has the flat top and bottom surface looks far from that in reality. This is mainly due to the flaw in model. To make the problem tractable, the model hypothesizes a flow induced by a moving body with the no slip boundary condition where the tangential derivative of the velocity potential is ignored, and the induced velocity is calculated to yield the pressure difference between the case when there is a slip on the surfaces and the case when there isn't. This is inconsistent with the irrotational flow analysis and results in the failure of describing the reality.

For more rigorous calculation, the flow field with a spherical bubble should be applied to determine the shape of a bubble, and the flow field is recalculated with the deformed shape. The recalculated flow field is used to give a new shape of a bubble. This iterative calculation between the flow field and the shape of the bubble is supposed to give the converged solutions to the velocity potential and the shape of a deformed bubble.

References

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Appendix A.

The functional expression for the free surface and its unit normal vector are

$$F = r - \eta(\phi) \tag{A.1}$$

$$\vec{n} = \frac{1}{\sqrt{1+\frac{1}{r^2}\left(\frac{\partial \eta}{\partial \phi}\right)^2}}\left[\vec{e}_r - \frac{1}{r}\left(\frac{\partial \eta}{\partial \phi}\right)\vec{e}_\phi\right] \tag{A.2}$$

By the coordination transformation, z-component of unit normal vector n_z is equal to:

$$n_z = \frac{1}{\sqrt{1+\frac{1}{r^2}\left(\frac{\partial \eta}{\partial \phi}\right)^2}}\left[\cos\phi + \frac{1}{r}\left(\frac{\partial \eta}{\partial \phi}\right)\sin\phi\right] \tag{A.3}$$

Using a spherical coordinate system, equation (1), (2) are rewritten as:

$$\nabla\Phi = \left(\frac{\partial \Phi}{\partial r}\right)\vec{e}_r + \frac{1}{r}\left(\frac{\partial \Phi}{\partial \phi}\right)\vec{e}_\phi = \vec{u} \tag{A.4}$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial \Phi}{\partial \phi}\right) = 0 \tag{A.5}$$

Equation (3), combined with equation (A.2), (A.3), (A.4), yields the expression for $\frac{\partial \Phi}{\partial \eta}$

at the surface $r = \eta(\phi)$

$$\frac{\partial \Phi}{\partial \eta} - \frac{1}{\eta^2}\frac{\partial \eta}{\partial \phi}\frac{\partial \Phi}{\partial \phi} = U_\infty\left[\cos\phi - \frac{1}{\eta}\left(\frac{\partial \eta}{\partial \phi}\right)\sin\phi\right] \tag{A.6}$$

Assuming no flow along the surface in the moving body with the no slip boundary.

$$d\Phi = \left(\frac{\partial \Phi}{\partial \eta}\right)d\eta + \left(\frac{\partial \Phi}{\partial \phi}\right)d\phi = 0 \tag{A.7}$$

By chain rule $\dfrac{\partial\Phi}{\partial\phi} = -\dfrac{\partial\eta}{\partial\phi}\dfrac{\partial\Phi}{\partial\eta}$, equation (A.6) is rearranged as:

$$\frac{\partial\Phi}{\partial\eta} = U_{\infty}\left[\cos\phi + \frac{1}{r}\left(\frac{\partial\eta}{\partial\phi}\right)\sin\phi\right]\cdot\left[1 + \frac{1}{r^2}\left(\frac{\partial\eta}{\partial\phi}\right)^2\right]^{-1} \qquad \text{(A.8)}$$

From equation (A.4) and the chain rule,

$$\left|\nabla\Phi\right|_s^2 = \left(\frac{\partial\Phi}{\partial\eta}\right)^2 + \frac{1}{\eta^2}\left(\frac{\partial\Phi}{\partial\phi}\right)^2 = \left(\frac{\partial\Phi}{\partial\eta}\right)^2\left[1 + \frac{1}{\eta^2}\left(\frac{\partial\Phi}{\partial\phi}\right)^2\right] \qquad \text{(A.9)}$$

On the other hand, the divergent of normal vector is equal to:

$$\nabla\cdot\vec{n} = \frac{2 + \dfrac{3}{\eta^2}\left(\dfrac{\partial\eta}{\partial\phi}\right)^2 - \left[\dfrac{1}{\eta}\left(\dfrac{\partial\eta}{\partial\phi}\right) + \dfrac{1}{\eta^3}\left(\dfrac{\partial\eta}{\partial\phi}\right)^3\right]\cot\phi - \dfrac{1}{\eta}\left(\dfrac{\partial^2\eta}{\partial\phi^2}\right)}{\eta\left[1 + \dfrac{1}{\eta^2}\left(\dfrac{\partial\eta}{\partial\phi}\right)^2\right]^{3/2}} \qquad \text{(A.10)}$$

Plugging equation (A.9), (A.10) into the equation (8) gives the equation (9) which is a non-linear 2nd ordinary differential equation.

Appendix B. Runge-Kutta-Nystrom Method

This algorithm computes the solution of the initial value problem

$$y'' = f(x, y, y')$$
$$y(x_0) = y_0$$
$$y'(x_0) = y'_0$$

at equidistant points $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, ….., $x_N = x_0 + Nh$; here f is such that this problem has a unique solution on the interval $[x_o, x_N]$.

INPUT: Initial values x_0, y_0, y'_0, step size h, number of steps N.

OUPUT: Approximation y_{n+1} to the solution $y(x_{n+1})$ at $x_{n+1} = x_0 + (n+1)h$, where

n = 0, 1, 2,…., N-1

For n = 0, 1, 2, …., N-1 do;

$$k_1 = \frac{1}{2} hf(x_n, y_n, y'_n)$$

$$k_2 = \frac{1}{2} hf(x_n + \frac{1}{2}h, y_n + K, y'_n + k_1) \quad \text{where } K = \frac{1}{2}h(y'_n + \frac{1}{2}k_1)$$

$$k_3 = \frac{1}{2} hf(x_n + \frac{1}{2}h, y_n + K, y'_n + k_2)$$

$$k_4 = \frac{1}{2} hf(x_n + h, y_n + L, y'_n + 2k_3) \quad \text{where } L = h(y'_n + k_3)$$

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + h(y'_n + \frac{1}{3}(k \ + k_2 + k_3))$$

OUTPUT x_{n+1}, y_{n+1}

$$y'_{n+1} = y'_n + \frac{1}{3}(k_1 + 2k_2 + 2k_3 + k_4)$$

End